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Control of mechanical systems with uncertain parameters by means of small forces☆

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ABSTRACT

An approach to the construction of a feedback control for non-linear Lagrange mechanical systems with uncertain parameters is developed. A Lagrange mechanical system with uncertain parameters, which is subject to the action of potential forces, control forces and unknown perturbations is considered is considered. It is assumed that the potential forces can be considerably greater than the control forces which, in their turn, are greater than the perturbations. An approach to the construction of a control, is proposed which enables one to bring a system from an arbitrary initial state to a specified final state in a finite time using a bounded control. A procedure, in which the specified nominal trajectory of the motion is tracked, is used. Initially, the trajectory, joining the specified initial and final states of the system, is constructed for a certain dynamical system which is close to the initial system but with completely known parameters. Then, using deviation equations, a control is constructed which brings the initial system onto this nominal trajectory in a finite time and subsequently forces the system to move along this nominal trajectory up to the final state. The control law used in tracking the nominal trajectory is based on a linear feedback, the gains of which depends on the discrepancy between the real trajectory and the nominal trajectory. The gain increase and tend to infinity as the discrepancies tend to zero but the control forces remain bounded and satisfy the conditions imposed on them. The results of numerical modelling of the controlled motions of a plane double pendulum are presented as an illustration.

In practice, problems of controlling various objects under conditions of incompleteness of the information concerning the objects themselves and the characteristics of the surrounding medium often arise. In particular, the geometric and mass-inertial parameters of an object may not be known (for example, in the case of a displacement of a load of unknown dimensions and mass by a manipulator). Uncontrolled perturbations acting on the system can emerge as another uncertain factor. Under these conditions, the role of the control algorithms, that allow the aims of the control to be attained in a finite time without violating the constraints imposed on the control actions, increases.

In recent years, new approaches to the construction of bounded controls for bringing mechanical systems into a specified final state in a finite time have been developed.

In one of these approaches,[1–5] the decomposition of a non-linear system with many degrees of freedom into simple, one-degree-of-freedom subsystems is used. The initial problem of the control of a non-linear system of order $2n$ is reduced to the problem of controlling a system of n simple independent second order linear equations. The methods of optimal control and differential games are used for each subsystem. The control obtained for the initial non-linear system as a result of this satisfies the constraints imposed and is close to optimal (suboptimal) control if the magnitudes of the perturbations and non-linearities in the system turn out to be small. In using this approach, a precise knowledge of the system kinetic energy matrix is required. Another approach has been used[6] for solving problems concerning the tracking of the trajectories of mechanical systems. This approach[7,8] is based on the stabilization of a specified preset trajectory of the system in phase space. A control law has been proposed[9] which ensures the strong asymptotic stability of a specified preset trajectory. This means that the current trajectory reaches the preset trajectory in a finite time, after which the system will move along the latter trajectory, that is, according to the specified program. In order to bring a system into the final state in a finite time using this approach, it is necessary to know the admissible preset trajectory (the admissibility of a trajectory means that, during the motion along this trajectory, the control

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forces satisfy the constraints imposed on them, whatever the admissible perturbations) which should start close to the initial state of the system and finish in the specified final state. One of the possible methods for constructing such a trajectory has been proposed[10].

Both the approaches presented above are applicable to scleronomic mechanical systems.

A further method[11,12] is based on Lyapunov's direct method and uses linear feedbacks with respect to the generalized coordinates and velocities with piecewise-constant coefficients (PD regulators). As the trajectory approaches the final state, the coefficients increase jumpwise and tend to infinity but the control forces remain bounded and satisfy the constraints imposed on them. This approach is also found to be applicable in the case of rheonomic mechanical systems[13], that is in the case of systems for which the kinetic energy is represented in the form of a complete quadratic polynomial of the generalized velocities with coefficients which depend explicitly on the generalized coordinates and time.

As a result of these methods, control laws are obtained which are described, generally speaking, by discontinuous functions, and moving sliding processes arise in the first two methods. An approach to constructing continuous feedback control laws, which ensure that a mechanical system is brought to the final state in a finite time, has been proposed[14,15]. It is also based on the Lyapunov's direct method. An implicitly specified Lyapunov function is used to construct the control law and its substantiation. The control law obtained in this case's an also be interpreted as a linear feedback but, here, the gains are smooth functions of the phase variables. These functions increase to infinity as the trajectory approaches the final state but the control forces remain bounded and satisfy the conditions imposed. This control method has been extended[16] to the case of rheonomic mechanical systems with unknown parameters.

The approaches enumerated above are effective on the assumption that the control forces are greater in magnitude than all the remaining forces acting on the system. This assumption is obviously limiting since, in many cases, a mechanical system turns out to be controlled using forces which are small compared with the other forces acting on it.[17] For example, a pendulum can be brought in a finite time into any final state by a force which is small compared with the gravitational force acting on it.

An approach, proposed earlier,[14] is used below to control a mechanical system with uncertain parameters on the assumption that the potential forces acting on the system are of greater magnitude than the control forces. For this purpose, a "nominal" trajectory, leading into the final state, is constructed for a certain "reference" system with completely known parameters. Then, using a trajectory tracking procedure, the initial system is brought along the nominal trajectory into the final state. This approach is found to be effective as applied to a system of deviation equations and enables a control law to be constructed for it which guarantees such behaviour of the initial system.

1. The basic control problem

A mechanical system is considered with dynamics described by Lagrange's equations of the second kind

$$\frac{d}{dt}\frac{\partial T}{\partial \dot{q}} - \frac{\partial T}{\partial q} = u + s - \frac{\partial P}{\partial q} \tag{1.1}$$

Here, $q, \dot{q} \in R^n$ are the vectors of the generalized coordinates and velocities, $T(q, \dot{q}) = \langle A(q)\dot{q}, \dot{q}\rangle/2$ is the kinetic energy of the system (henceforth $\langle \cdot,\cdot \rangle$ denotes a scalar product), u is the vector of the generalized control forces, s is the vector of the unknown generalized forces which we shall call perturbations and $P(q) \in C^1$ is the potential energy of the system. It is assumed that the function $P(q)$ has a global minimum. Without loss in generality, we shall assume this minimum of the function $P(q)$ is attained at the point $q = 0$.

We will assume that the mass-inertial parameters of the system are imprecisely known, that is, elements of the kinetic energy matrix $A(q)$ and the function $P(q)$ are unknown but lie within specified limits. This means that the positive-definite, continuously differentiable, symmetric matrix $A(q)$ can be represented in the form

$$A(q) = A_0(q) + A_1(q) \tag{1.2}$$

Here, $A_0(q)$, $A_1(q)$ are symmetric, continuously differentiable matrices, and the matrix $A_0(q)$ is positive-definite and known but $A_1(q)$ is an unknown matrix.

Correspondingly, the potential energy of the system can be represented in the form

$$P(q) = P_0(q) + P_1(q) \tag{1.3}$$

where $P_0(q)$ is a specified function and $P_1(q)$ is an unknown function. We shall assume that the function $P_0(q)$ also has a global minimum at the point $q = 0$.

Assume that

$$T_0(q,\dot{q}) = \langle A_0(q)\dot{q},\dot{q}\rangle/2, \quad T_1(q,\dot{q}) = \langle A_1(q)\dot{q},\dot{q}\rangle/2$$

Then, taking account of equality (1.2), we have

$$T(q,\dot{q}) = T_0(q,\dot{q}) + T_1(q,\dot{q})$$

It is assumed that the eigenvalues of the matrices $A(q)$ and $A_0(q)$ lie in the segment $[m, M]$, $0 < m \le M$ for any q, that is,

$$mz^2 \le \langle A(q)z,z\rangle \le Mz^2, \quad mz^2 \le \langle A_0(q)z,z\rangle \le Mz^2, \quad \forall q,z \in R^n \tag{1.4}$$

and the matrix A_1 and the partial derivatives of the matrices A_0, A_1 and A are uniformly bounded according to the norm ($i, j = 1,\ldots,n$):

$$\|A_1(q)\| \le M_1, \quad \left\|\frac{\partial A}{\partial q_i}(q)\right\| \le C, \quad \left\|\frac{\partial A_0}{\partial q_i}(q)\right\| \le C_0$$

$$\left\|\frac{\partial A_1}{\partial q_i}(q)\right\| \le C_1, \quad \left\|\frac{\partial^2 A}{\partial q_i \partial q_j}(q)\right\| \le C_2, \quad M_1, C, C_0, C_1, C_2 > 0 \tag{1.5}$$

Henceforth, the Euclidean norm of a vector z is denoted by $|z|$, its scalar square $|z^2|$ is denoted by $\langle z, z\rangle$ and the Euclidean norm of the matrix Z, which is understood as the norm of the corresponding linear operator in Euclidean space, is denoted by $|Z|$.

We shall assume that, in the case of the partial derivatives of the functions $P(q)$ and $P_1(q)$, the following inequalities are satisfied

$$\left|\frac{\partial P_1}{\partial q}(q)\right| \le D_1, \quad \left\|\frac{\partial^2 P}{\partial q^2}(q)\right\| \le D_2 \tag{1.6}$$

The constraint

$$|u| \le u_0, \ u_0 > 0 \tag{1.7}$$

is imposed on the vector of the control forces.

The vector of the unknown perturbations $s(t, q, \dot{q})$ can be an arbitrary vector function, including a discontinuous function, which satisfies some conditions for the existence of a solution of system (1.1) and the condition

$$|s(t, q, \dot{q})| \le s_0, \ s_0 > 0 \tag{1.8}$$

It is assumed that the perturbations s are small compared with the control forces u, that is, $u_0 > s_0$. This inequality will be refined below.

The phase coordinates and velocities q, $\dot{q}$ are assumed to be accessible to measurement at each instant.

We denote the initial state by $(q_0, \dot{q}_0)$ and the final state of the system by $(q_*, \dot{q}_*)$ and consider the problem of constructing a control u which satisfies condition (1.7) and which transfers system (1.1) from the state $(q_0, \dot{q}_0)$ into the state $(q_*, \dot{q}_*)$ in a finite time.

Together with system (1.1), we shall consider the unperturbed system with completely known parameters

$$\frac{d}{dt}\frac{\partial T_0}{\partial \dot{q}} - \frac{\partial T_0}{\partial q} = u' - \frac{\partial P_0}{\partial q} \tag{1.9}$$

We make the following simplifying assumption.

Assumption. We shall assume that the control laws bringing systems (1.1) and (1.9) into a small neighbourhood of the origin of the phase space coordinates and satisfying condition (1.7) are known for any initial states.

Remark 1. To justify this assumption, we note that the origin of the phase space coordinates when there are no control forces and perturbations, that is, when $u = u' = s = 0$, is a stable stationary state both for system (1.1) and for system (1.9), by virtue of the fact that the potential energy of these systems has a global minimum at the point $q = 0$. This fact enables as to use, for example, a control directed against the velocity:

$$u = -\frac{u_0}{|\dot{q}|}\dot{q}, \quad \dot{q} \ne 0$$

to bring the system into the neighbourhood of the origin of coordinates.

Such a control is also found to be effective when there are perturbations if $u_0 > s_0$. This control will be used below to control a double pendulum.

The problem is solved in two stages. In the first stage, system (1.1) is brought in a finite (unfixed) time into a small neighbourhood of the origin of the phase space coordinates and, in the second stage, from this neighbourhood into the final state. In accordance with the assumption, the control that provides transferring in the first stage is known. We therefore immediately turn to the second stage, reformulating the problem in the following manner.

Problem 1. Suppose the initial state of system $(q_0, \dot{q}_0)$ lies in a small neighbourhood of the origin of the phase space coordinates $q = \dot{q} = 0$. It is required to construct a control u, satisfying constraint (1.7), which brings system (1.1) into the state $(q_*, \dot{q}_*)$ in a finite time, no matter what the matrix $A(q)$, the function $P(q)$ or the perturbations s are, provided that they satisfy the conditions imposed on them.

2. Construction of the nominal trajectory

We will use the trajectory tracking method to solve Problem 1. For this purpose, we construct the "nominal" trajectory leading to the final state for the unperturbed system (1.9).

According to the assumption, the control u' which brings system (1.9) from the point $(q_*, \dot{q}_*)$ (that is identical to the final state of the perturbed system (1.1)) to a certain point $q_*^0, \dot{q}_*^0$, lying in a small neighbourhood of zero, is known. At the same time, this control can be chosen as satisfying the condition

$$|u'| \le u_0/2 \tag{2.1}$$

Consequently, the control transferring system (1.9) in the reverse direction, that is, from the point $q_*^0, \dot{q}_*^0$ to the point $q_*, \dot{q}_*$, is known Actually, the following symmetry property can be used to construct such a control: if a control in the form of a feedback $u(q, \dot{q})$ (a preset control $u(t)$, $t\in[0, T]$ respectively) brings system (1.9) from a state $(q^1, \dot{q}^1)$ into a state $(q^2, \dot{q}^2)$, then the control $u(q, -\dot{q})$ $(u(T-t))$ brings the system from a state $(q^2, \dot{q}^2)$ into a state $(q^1, \dot{q}^1)$.

The control which transfers system (1.9) from the point $q_*^0, \dot{q}_*^0$ to the point $q_*, \dot{q}_*$ and satisfies condition (2.1) can therefore be considered as being known. We will denote this control by u'. We will call the trajectory of system (1.9), joining the point $q_*^0, \dot{q}_*^0$ and the final state $q_*, \dot{q}_*$, the nominal trajectory and we will denote it by $\tilde{q}(t), \dot{\tilde{q}}(t)$.

The instant when the second stage of the motion commences is denoted by t_0. Then,

$$\tilde{q}(t_0) = q_*^0, \quad \dot{\tilde{q}}(t_0) = \dot{q}_*^0$$

We now construct the control u'' which satisfies the constraint

$$|u''| \le u_0/2 \tag{2.2}$$

and brings the initial system with the perturbations (1.1) from the point $q_*, \dot{q}_*$ onto the nominal trajectory and keeps the system on this trajectory until it arrives at the final state.

We will now consider the deviation equations for this.

3. Deviation equations

We rewrite system (1.1) in the form

$$\sum_j a_{ij}(q)\ddot{q}_j = -\sum_{j,k}\left(\frac{\partial a_{ij}}{\partial q_k}(q) - \frac{1}{2}\frac{\partial a_{jk}}{\partial q_i}(q)\right)\dot{q}_j\dot{q}_k + s_i + u_i' + u_i'' + \frac{\partial P}{\partial q_i}(q) \tag{3.1}$$

where a_{ij} $(i, j=1,\ldots,n)$ are the elements of the kinetic energy matrix of the system $A(q)$, u', u'' are the control force vectors and

$$u' + u'' = u$$

Henceforth, summation with respect to j, k and r is carried out from 1 to n.

Note that the control function u will satisfy condition (1.7) since constraints (2.1) and (2.2) are imposed on the functions u' and u''.

The deviations of the phase coordinates and velocities of the trajectory of the perturbed system (1.1) from the nominal trajectory of the unperturbed system (1.9) is denoted by x, $\dot{x}$, that is,

$$x(t) = q(t) - \tilde{q}(t), \quad \dot{x}(t) = \dot{q}(t) - \dot{\tilde{q}}(t)$$

Since the perturbed system (1.1) at the instant to when the second stage commences is in the state $(q_0, \dot{q}_0)$, the initial deviation of the trajectory of the system from the nominal trajectory can be written in the form

$$x_0 = x(t_0) = q_0 - q_*^0, \quad \dot{x}_0 = \dot{x}(t_0) = \dot{q}_0 - \dot{q}_*^0$$

To solve Problem 1(and, consequently, the main control problem), it is sufficient to solve the following problem.

Problem 2. It is required to construct a control u'' which satisfies constraint (2.2) and to determine the domain $X\subset R^{2n}$ of admissible initial deviations $x_0, \dot{x}_0$ such that any trajectory of system (3.1) starting in this domain will reach the nominal trajectory in a finite time, and the system will subsequently move along this trajectory whatever the perturbations s satisfying constraint (1.8).

The elements of the matrices A_0 and A_1 are denoted by a_{ij}^0 and a_{ij}^1 respectively. Then,

$$a_{ij} = a_{ij}^0 + a_{ij}^1 \tag{3.2}$$

We rewrite the equation of motion of system (1.9) along the nominal trajectory in the form

$$\sum_j a_{ij}^0(\tilde{q})\ddot{\tilde{q}}_j = -\sum_{j,k}\left(\frac{\partial a_{ij}^0}{\partial q_k}(\tilde{q}) - \frac{1}{2}\frac{\partial a_{jk}^0}{\partial q_i}(\tilde{q})\right)\dot{\tilde{q}}_j\dot{\tilde{q}}_k + u_i' + \frac{\partial P_0}{\partial q_i}(\tilde{q}) \tag{3.3}$$

and the equations of motion of the perturbed system in the form

$$\sum_j a_{ij}(\tilde{q}+x)(\ddot{\tilde{q}}_j + \ddot{x}_j) = -\sum_{j,k}^n\left(\frac{\partial a_{ij}}{\partial q_k}(\tilde{q}+x) - \frac{1}{2}\frac{\partial a_{jk}}{\partial q_i}(\tilde{q}+x)\right)(\dot{\tilde{q}}_j + \dot{x}_j)(\dot{\tilde{q}}_k + \dot{x}_k) + $$
$$+ s_i + u_i' + u_i'' + \frac{\partial P}{\partial q_i}(\tilde{q}+x) \tag{3.4}$$

Applying Taylor's formula with the residual term in Lagrange form

$$a_{ij}(\tilde{q}+x) = a_{ij}(\tilde{q}) + \sum_k \frac{\partial a_{ij}}{\partial q_k}(q')x_k$$

where $q' = \tilde{q} + \theta' x$, $0 < \theta' < 1$, we transform the left-hand side of equality (3.4) to the form

$$\sum_j a_{ij}(\tilde{q}+x)(\ddot{\tilde{q}}_j + \ddot{x}_j) = \sum_j a_{ij}(\tilde{q}+x)\ddot{x}_j + \sum_j \left(a_{ij}^0(\tilde{q}) + a_{ij}^1(\tilde{q})\right)\ddot{\tilde{q}}_j + \sum_{j,k}^n \frac{\partial a_{ij}}{\partial q_k}(q')x_k\ddot{\tilde{q}}_j \tag{3.5}$$

We introduce the notation

$$\Gamma_{ijk}(q) = \frac{\partial a_{ij}}{\partial q_k}(q) - \frac{1}{2}\frac{\partial a_{jk}}{\partial q_i}(q)$$

$$\Gamma_{ijk}^p(q) = \frac{\partial a_{ij}^p}{\partial q_k}(q) - \frac{1}{2}\frac{\partial a_{jk}^p}{\partial q_i}(q), \quad p = 0,1$$

Using the equalities

$$\Gamma_{ijk}(\tilde{q}+x) = \Gamma_{ijk}(\tilde{q}) + \sum_{r=1}^n \frac{\partial \Gamma_{ijk}}{\partial q_r}(q'')x_r, \quad q'' = \tilde{q} + \theta'' x, \quad 0 < \theta'' < 1$$

$$\frac{\partial P}{\partial q_i}(\tilde{q}+x) = \frac{\partial P}{\partial q_i}(\tilde{q}) + \sum_{j=1}^n \frac{\partial^2 P}{\partial q_i \partial q_j}(q''')x_j, \quad q''' = \tilde{q} + \theta''' x, \quad 0 < \theta''' < 1$$

we reduce the expression on the right-hand side of equality (3.4) to the form

$$\begin{aligned}
&\sum_{j,k}\Gamma_{ijk}(\tilde{q}+x)\left(\dot{\tilde{q}}_j\dot{\tilde{q}}_k + \dot{\tilde{q}}_j\dot{x}_k + \dot{\tilde{q}}_k\dot{x}_j + \dot{x}_j\dot{x}_k\right) = \\
&= \sum_{j,k}\left(\Gamma_{ijk}^0(\tilde{q}) + \Gamma_{ijk}^1(\tilde{q})\right)\dot{\tilde{q}}_j\dot{\tilde{q}}_k + \sum_{j,k}\Gamma_{ijk}(\tilde{q}+x)\left(\dot{\tilde{q}}_j\dot{x}_k + \dot{\tilde{q}}_k\dot{x}_j + \dot{x}_j\dot{x}_k\right) + \\
&+ \sum_{j,k,r}\frac{\partial \Gamma_{ijk}}{\partial q_r}(q'')x_r\dot{\tilde{q}}_j\dot{\tilde{q}}_k + s_i + u_i' + u_i'' + \frac{\partial P_0}{\partial q_i}(\tilde{q}) + \frac{\partial P_1}{\partial q_i}(\tilde{q}) + \sum_j \frac{\partial^2 P}{\partial q_i \partial q_j}(q'')x_j
\end{aligned} \tag{3.6}$$

Taking relations (3.2)- (3.6) into account, we write the deviation equations as follows:

$$\begin{aligned}
&\sum_j a_{ij}(\tilde{q}+x)\ddot{x}_j = -\sum_j a_{ij}^1(\tilde{q})\ddot{\tilde{q}}_j - \sum_{j,k}\frac{\partial a_{ij}}{\partial q_k}(q')x_k\ddot{\tilde{q}}_j - \sum_{j,k}\Gamma_{ijk}^1(\tilde{q})\dot{\tilde{q}}_j\dot{\tilde{q}}_k - \\
&- \sum_{j,k,r}\frac{\partial \Gamma_{ijk}}{\partial q_r}(q'')x_r\dot{\tilde{q}}_j\dot{\tilde{q}}_k - \sum_{j,k}\Gamma_{ijk}(\tilde{q}+x)\left(\dot{\tilde{q}}_j\dot{x}_k + \dot{\tilde{q}}_k\dot{x}_j + \dot{x}_j\dot{x}_k\right) + \\
&+ s_i + u_i'' + \frac{\partial P_1}{\partial q_i}(\tilde{q}) + \sum_j \frac{\partial^2 P}{\partial q_i \partial q_j}(q''')x_j
\end{aligned}$$

or, in vector form,

$$A(\tilde{q}+x)\ddot{x} = u'' + S \tag{3.7}$$

where

$$S = -A_1(\tilde{q})\ddot{\tilde{q}} + F_1 + F_2 + \ldots + F_{11} + s \tag{3.8}$$

Here,

$$F_1 = -\left(\sum_k \frac{\partial A}{\partial q_k}(q')x_k\right)\ddot{\tilde{q}}, \quad F_2 = -\left(\sum_k \frac{\partial A_1}{\partial q_k}(\tilde{q})\dot{\tilde{q}}_k\right)\dot{\tilde{q}}, \quad F_3 = \frac{1}{2}\frac{\partial}{\partial q}\left\langle A_1(\tilde{q})\dot{\tilde{q}}, \dot{\tilde{q}}\right\rangle$$

$$F_4 = -\left(\sum_{k,r}\frac{\partial^2 A}{\partial q_k \partial q_r}(q'')x_r\dot{\tilde{q}}_k\right)\dot{\tilde{q}}, \quad F_5 = \frac{1}{2}\frac{\partial}{\partial q}\left\langle\left(\sum_r \frac{\partial A}{\partial q_r}(q'')x_r\right)\dot{\tilde{q}}, \dot{\tilde{q}}\right\rangle$$

$$F_6 = \frac{\partial}{\partial q}\left(\left\langle A(\tilde{q}+x)\dot{\tilde{q}}, \dot{x}\right\rangle + \frac{1}{2}\left\langle A(\tilde{q}+x)\dot{x}, \dot{x}\right\rangle\right), \quad F_7 = -\left(\sum_k \frac{\partial A}{\partial q_k}(\tilde{q}+x)\dot{x}_k\right)\dot{\tilde{q}}$$

$$F_8 = -\left(\sum_k \frac{\partial A}{\partial q_k}(\tilde{q}+x)\dot{\tilde{q}}_k\right)\dot{x}, \quad F_9 = -\left(\sum_k \frac{\partial A}{\partial q_k}(\tilde{q}+x)\dot{x}_k\right)\dot{x}, \; F_{10} = \frac{\partial P_1}{\partial q}(\tilde{q}), \quad F_{11} = \frac{\partial^2 P}{\partial q^2}(q''')x$$

We now estimate the individual terms in expression (3.8) for S. We shall assume that the phase coordinates, velocities and accelerations along the nominal trajectory satisfy the constraints

$$|\tilde{q}| \le Q, \quad |\dot{\tilde{q}}| \le Q_1, \quad |\ddot{\tilde{q}}| \le Q_2 \tag{3.9}$$

It follows from these relations and the first inequality of (1.5) that

$$|A_1(\tilde{q})\ddot{\tilde{q}}| \le M_1 Q_2 \tag{3.10}$$

By virtue of inequalities (1.5) and (3.9) and the inequality

$$\sum_i |z_1|+|z_2|+\ldots+|z_n| \le \sqrt{n}|z|, \quad z \in R^n \tag{3.11}$$

the estimates

$$|F_1| \le C\sum_k |x_k||\ddot{\tilde{q}}| \le \sqrt{n}CQ_2|x|, \quad |F_2| \le C_1\sum_k |\dot{\tilde{q}}_k||\dot{\tilde{q}}| \le \sqrt{n}C_1Q_1^2$$

$$|F_7| \le C\sum_k |\dot{x}_k||\dot{\tilde{q}}| \le \sqrt{n}CQ_1|\dot{x}|, \quad |F_8| \le C\sum_k |\dot{\tilde{q}}_k||\dot{x}| \le \sqrt{n}CQ_1|\dot{x}|$$

$$|F_9| \le C\sum_k |\dot{x}_k||\dot{x}| \le \sqrt{n}C|\dot{x}|^2 \tag{3.12}$$

hold.

The relations

$$\left|\frac{\partial}{\partial q_i}\langle A_1(\tilde{q})\dot{\tilde{q}},\dot{\tilde{q}}\rangle\right| \le C_1|\dot{\tilde{q}}|^2 \le C_1Q_1^2$$

$$\left|\frac{\partial}{\partial q_i}\left\langle\left(\sum_r \frac{\partial A}{\partial q_r}(q'')x_r\right)\dot{\tilde{q}},\dot{\tilde{q}}\right\rangle\right| = \left|\left\langle\left(\sum_r \frac{\partial^2 A}{\partial q_i \partial q_r}(q'')x_r\right)\dot{\tilde{q}},\dot{\tilde{q}}\right\rangle\right| \le$$

$$\le C_2\sum_r |x_r||\dot{\tilde{q}}|^2 \le \sqrt{n}C_2Q_1^2|x|, \quad \left|\frac{\partial}{\partial q_i}\langle A(\tilde{q}+x)\dot{\tilde{q}},\dot{x}\rangle\right| \le C|\dot{\tilde{q}}||\dot{x}| \le CQ_1|\dot{x}|$$

$$\left|\frac{\partial}{\partial q_i}\langle A(\tilde{q}+x)\dot{x},\dot{x}\rangle\right| \le C|\dot{x}|^2, \quad i = 1,\ldots,n$$

follow from conditions (1.5) and (3.9), inequality (3.11) and the Cauchy inequality, and, from these relations, using inequality (3.11), we obtain

$$|F_3| \le \frac{\sqrt{n}}{2}C_1Q_1^2, \quad |F_5| \le \frac{n}{2}C_2Q_1^2|x|, \quad |F_6| \le \sqrt{n}C|\dot{x}|\left(Q_1 + \frac{|\dot{x}|}{2}\right) \tag{3.13}$$

Moreover, we have

$$|F_4| \le |\dot{\tilde{q}}|\sum_{k,r} C_2|x_r\dot{\tilde{q}}_k| \le \sqrt{n}C_2|x||\dot{\tilde{q}}|\sum_k |\dot{\tilde{q}}_k| \le nC_2\,|\,\dot{\tilde{q}}|^2|x| \le nC_2Q_1^2|x| \tag{3.14}$$

Using relations (1.6) and (3.9), we estimate the remaining terms in relation (3.8) in the following manner

$$|F_{10}| \le D_1, \quad |F_{11}| \le D_2|x| \tag{3.15}$$

The estimate

$$|S| \le S_0 \tag{3.16}$$

where

$$S_0 = M_1Q_2 + D_1 + s_0 + \frac{3}{2}\sqrt{n}C_1Q_1^2 +$$

$$+\left(\sqrt{n}CQ_2 + \frac{3}{2}nC_2Q_1^2 + D_2\right)|x| + 3\sqrt{n}C\left(Q_1 + \frac{|\dot{x}|}{2}\right)|\dot{x}|$$

follows from relations (1.8), (3.10) and (3.12) - (3.15).

We now construct the control $u''(t, x, \dot{x})$ in the form of a feedback, which satisfies constraint (2.2) and which brings system (3.7) from any initial state to the origin of coordinates $x = \dot{x} = 0$ in a finite time. This will signify that the control $u = u' + u''$ satisfies constraint (1.7) and brings the initial system with perturbations (1.1) from the point $(q_0, \dot{q}_0)$ onto the nominal trajectory in a finite time and keeps the system on this trajectory until it arrives at final state.

4. Auxiliary control problem

We introduce the notation

$$\bar{A}(t,x) = A(\tilde{q}(t) + x), \quad \bar{A}_i(t,x) = A_i(\tilde{q}(t) + x)$$

$$\bar{T}_i(t,x,\dot{x}) = \langle \bar{A}_i(t,x)\dot{x},\dot{x}\rangle/2, \quad i = 0,1$$

Then,

$$\bar{A}(t,x) = \bar{A}_0(t,x) + \bar{A}_1(t,x), \quad \bar{T}(t,x,\dot{x}) = \bar{T}_0(t,x,\dot{x}) + \bar{T}_1(t,x,\dot{x})$$

We now formulate an auxiliary control problem and, to do this, we consider a control system, the dynamics of which are described by the equations

$$\bar{A}(t,x)\ddot{x} = u'' + S \tag{4.1}$$

It follows from assumptions (1.4) that the matrices $\bar{A}$ and $\bar{A}_0$ satisfy the relations

$$mz^2 \le \langle \bar{A}(t,x)z,z\rangle \le Mz^2, \quad mz^2 \le \langle \bar{A}_0(t,x)z,z\rangle \le Mz^2 \tag{4.2}$$

for any x and t, and, by virtue of assumptions (1.5), the matrix $\bar{A}_1$ and the partial derivatives of the matrices $\bar{A}_0, \bar{A}_1$ and $\bar{A}$ are uniformly bounded with respect to the norm $(i, j = 1,\ldots,n)$:

$$\|\bar{A}_1(t,x)\| \le M_1, \quad \left\|\frac{\partial \bar{A}}{\partial x_i}(t,x)\right\| \le C, \quad \left\|\frac{\partial \bar{A}_0}{\partial x_i}(t,x)\right\| \le C_0$$

$$\left\|\frac{\partial \bar{A}_1}{\partial x_i}(t,x)\right\| \le C_1, \quad \left\|\frac{\partial^2 \bar{A}}{\partial x_i \partial x_j}(t,x)\right\| \le C_2, \quad M_1, C, C_0, C_1, C_2 > 0 \tag{4.3}$$

Since

$$\frac{\partial \bar{A}_0}{\partial t}(t,x) = \frac{\partial A_0}{\partial q}(\tilde{q}(t) + x))\dot{\tilde{q}}(t)$$

the inequality

$$\left\|\frac{\partial \bar{A}_0(t,x)}{\partial t}\right\| \le C_0 Q_1 \tag{4.4}$$

follows from assumptions (1.5) and (3.9).

The control vector function $u''(t, x, \dot{x})$ satisfies the condition

$$|u''(t,x,\dot{x})| \le U_0, \quad U_0 = u_0/2 \tag{4.5}$$

and the unknown vector function $S(t, x, \dot{x})$ satisfies the condition

$$|S(t,x,\dot{x})| \le S_0 \tag{4.6}$$

We now introduce the domain of the extended phase space

$$D = \left\{(t,x,\dot{x}) \in R^{2n+1} : x^2 + \dot{x}^2 \ne 0\right\}$$

Problem 3. It is required to construct a control $u''(t, x, \dot{x})$ as a vector function of the phase variables x, $\dot{x}$ and the time in the domain D which satisfies condition (4.5) and to determine a set of admissible initial states such that any trajectory of system (4.1) beginning in this set arrives at the origin of coordinates of the phase space $x = \dot{x} = 0$ in a finite time, no matter what the matrix $\bar{A}$ and the function S, satisfying conditions (4.2) - (4.4) and (4.6) respectively, are.

5. Control for a system of deviation equations

We will now determine the required control in the domain D in the form

$$u''(t,x,\dot{x}) = -a(t,x,\dot{x})\bar{A}_0(t,x)\dot{x} - b(t,x,\dot{x})x \tag{5.1}$$

where

$$a(t,x,\dot{x}) = \sqrt{\frac{b(t,x,\dot{x})}{M}}, \quad b(t,x,\dot{x})\frac{3U_0^2}{8V(t,x,\dot{x})} \tag{5.2}$$

$$V(t,x,\dot{x}) = \overline{T}_0 + \frac{1}{2}b(t,x,\dot{x})x^2 + \frac{1}{2}a(t,x,\dot{x})\langle \overline{A}_0(t,x)\dot{x},x\rangle \tag{5.3}$$

Relations (5.2) and (5.3) specify the functions $a(t, x, \dot{x})$, $b(t, x, \dot{x})$ and $V(t, x, \dot{x})$ in an implicit manner.

It has been shown[15] that continuously differentiable positive functions $a(t, x, \dot{x})$, $b(t, x, \dot{x})$ and $V(t, x, \dot{x})$ exist in the domain D which satisfy these relations. It has been established that the control function $u''(t, x, \dot{x})$ (5.1) satisfies constraint (4.5) and inequalities have been obtained for the functions $V(t, x, \dot{x})$:

$$V_-(t,x,\dot{x}) \le V(t,x,\dot{x}) \le 3V_-(t,x,\dot{x}) \tag{5.4}$$

where

$$V_-(t,x,\dot{x}) = \frac{1}{4}\left(2\overline{T}_0(t,x,\dot{x}) + b(t,x,\dot{x})x^2\right) \tag{5.5}$$

The following two-sided estimates of the function $V(t, x, \dot{x})$, specified implicitly in terms of the phase variables $x, \dot{x}$ and the known parameters of the problem m, M and U_0, follow from these relations:

$$\frac{1}{8}\left(m\dot{x}^2 + \left(m^2\dot{x}^4 + 6U_0^2x^2\right)^{1/2}\right) \le V(t,x,\dot{x}) \le \frac{3}{8}\left(M\dot{x}^2 + \left(M^2\dot{x}^4 + 2U_0^2x^2\right)^{1/2}\right) \tag{5.6}$$

Estimates (5.6) show that the function V is positive-definite and allows of an infinitesimal upper limit. It is easy to see that the function V can be defined as zero in the set $\{(t, 0, 0) \in R^{2n+1}, t \ge 0\}$ with preservation of continuity. However, it will not be differentiable at the points of this set.

We now find the domain of the admissible states and show that any trajectory of system (4.1) starting in this domain, which is controlled according to the law (5.1)- (5.3), arrives at the origin of coordinates in a finite time. To do this, we will use methods of the theory of stability and show that the function V is Lyapunov's function of the system considered. For this purpose, we calculate the derivative $\dot{V}$.

We differentiate the functions $a(t, x, \dot{x})$, $b(t, x, \dot{x})$ and $V(t, x, \dot{x})$ and obtain

$$\dot{a} = -\frac{a}{2V}\dot{V}, \quad \dot{b} = -\frac{b}{V}\dot{V}$$

$$\dot{V} = \dot{T}_0 + b\langle x,\dot{x}\rangle + a\overline{T}_0 + \frac{a}{2}\left\langle \frac{d}{dt}(\overline{A}_0\dot{x}),x\right\rangle - \frac{\dot{V}}{2V}\left(bx^2 + \frac{a}{2}\langle \overline{A}_0\dot{x},x\rangle\right) \tag{5.7}$$

Taking account of the relations

$$\langle \overline{A}_0\dot{x},\ddot{x}\rangle = \langle \overline{A}_0\ddot{x},\dot{x}\rangle, \quad \overline{A}_0\ddot{x} = (\overline{A} - \overline{A}_1)\ddot{x} = u'' + S - \overline{A}_1\ddot{x} = -a\overline{A}_0\dot{x} - bx + S - \overline{A}_1\ddot{x}$$

we have

$$\frac{d}{dt}(\overline{A}_0\dot{x}) = \overline{A}_0\ddot{x} + \frac{\partial \overline{A}_0}{\partial t}\dot{x} + \left(\sum_k \frac{\partial \overline{A}_0}{\partial x_k}\dot{x}_k\right)\dot{x} = -a\overline{A}_0\dot{x} - bx + S - \overline{A}_1\ddot{x} + \frac{\partial \overline{A}_0}{\partial t}\dot{x} + \left(\sum_k \frac{\partial \overline{A}_0}{\partial x_k}\dot{x}_k\right)\dot{x} \tag{5.8}$$

We now calculate the derivative of the function $\bar{T}_0(t, x, \dot{x})$ by virtue of system (4.1), (5.1):

$$\dot{\overline{T}}_0 = \frac{1}{2}\left\langle \frac{d}{dt}(\overline{A}_0\dot{x}) + \overline{A}_0\ddot{x},\dot{x}\right\rangle = -2a\overline{T}_0 - b\langle x,\dot{x}\rangle + $$
$$+ \langle S,\dot{x}\rangle - \langle \overline{A}_1\ddot{x},\dot{x}\rangle + \frac{1}{2}\left\langle \frac{\partial \overline{A}_0}{\partial t}\dot{x},\dot{x}\right\rangle + \frac{1}{2}\left\langle \left(\sum_k \frac{\partial \overline{A}_0}{\partial x_k}\dot{x}_k\right)\dot{x},\dot{x}\right\rangle \tag{5.9}$$

We substitute expressions (5.8) and (5.9) into expression (5.7) for $\dot{V}$ and we obtain

$$\dot{V} = -a\left(\overline{T}_0 + \frac{b}{2}x^2 + \frac{a}{2}\langle \overline{A}_0\dot{x},x\rangle\right) - \frac{\dot{V}}{2V}\left(bx^2 + \frac{a}{2}\langle \overline{A}_0\dot{x},x\rangle\right) + $$
$$+ \left\langle S - \overline{A}_1\ddot{x},\dot{x} + \frac{a}{2}x\right\rangle + \frac{1}{2}\left\langle \left(\frac{\partial \overline{A}_0}{\partial t} + \sum_k \frac{\partial \overline{A}_0}{\partial x_k}\dot{x}_k\right)\dot{x},\dot{x} + ax\right\rangle \tag{5.10}$$

The equality

$$a\left(\overline{T}_0 + \frac{b}{2}x^2 + \frac{a}{2}\langle \overline{A}_0\dot{x},x\rangle\right) = aV = \frac{\sqrt{3}U_0}{2\sqrt{2M}}V^{1/2} \tag{5.11}$$

follows from definitions (5.2) and (5.3) of the functions a and V.

We now put

$$B(t,x,\dot{x}) = 1 + \frac{b}{2V}x^2 + \frac{a}{4V}\langle \bar{A}_0\dot{x},x\rangle = \frac{1}{V}\left(\bar{T}_0 + bx^2 + \frac{3a}{4}\langle \bar{A}_0\dot{x},x\rangle\right) \tag{5.12}$$

Substituting expressions (5.11) and (5.12) into equality (5.10), we obtain the final expression for the derivative function $V(t, x, \dot{x})$ by virtue of system (4.1), (5.1):

$$B\dot{V} = -\frac{\sqrt{3}U_0}{2\sqrt{2M}}V^{1/2} + \left\langle S - \bar{A}_1\ddot{x}, \dot{x} + \frac{a}{2}x\right\rangle + \frac{1}{2}\left\langle\left(\frac{\partial\bar{A}_0}{\partial t} + \sum_k\frac{\partial\bar{A}_0}{\partial x_k}\dot{x}_k\right)\dot{x}, \dot{x} + ax\right\rangle \tag{5.13}$$

We now estimate the individual terms in the expression for $\dot{V}$. By virtue of the Cauchy inequality, conditions (4.2) and relations (5.2), (5.4) and (5.5), we have

$$|\dot{x} + \frac{a}{2}x|^2 \le \frac{5}{4}(\dot{x}^2 + a^2x^2) \le \frac{5}{4}\left(\frac{2}{m}\bar{T}_0 + \frac{b}{M}x^2\right) \le \frac{5}{m}V_- \le \frac{5}{m}V$$

$$|\dot{x} + ax|^2 \le 2(\dot{x}^2 + a^2x^2) \le 2\left(\frac{2}{m}\bar{T}_0 + \frac{b}{M}x^2\right) \le \frac{8}{m}V_- \le \frac{8}{m}V$$

$$\dot{x}^2 \le \frac{2}{m}\bar{T}_0 \le \frac{4}{m}V_- \le \frac{4}{m}V \tag{5.14}$$

The inequality

$$\left|\left\langle S, \dot{x} + \frac{a}{2}x\right\rangle\right| \le \sqrt{\frac{5}{m}}S_0V^{1/2} \tag{5.15}$$

follows from here and from condition (4.6).

By virtue of (4.2), (4.3), (4.5) and (4.6), the relations

$$|\bar{A}_1\ddot{x}| = |\bar{A}_1\bar{A}^{-1}\bar{A}\ddot{x}||\bar{A}_1\bar{A}^{-1}(u'' + S)| \le \frac{M_1}{m}(U_0 + S_0)$$

hold, whence, taking inequalities (5.14) into account, we obtain

$$\left|\left\langle \bar{A}_1\ddot{x}, \dot{x} + \frac{a}{2}x\right\rangle\right| \le \frac{M_1\sqrt{5}}{m\sqrt{m}}(U_0 + S_0)V^{1/2} \tag{5.16}$$

The estimates

$$\left|\frac{1}{2}\left\langle\frac{\partial\bar{A}_0}{\partial t}\dot{x}, ax + \dot{x}\right\rangle\right| \le \frac{2\sqrt{2}}{m}C_0Q_1V$$

$$\left|\frac{1}{2}\left\langle\sum_k\frac{\partial\bar{A}_0}{\partial x_k}\dot{x}_k\dot{x}, \dot{x} + ax\right\rangle\right| \le \frac{1}{2}\sqrt{n}C_0\dot{x}^2|\dot{x} + ax| \le \frac{4\sqrt{2n}}{m\sqrt{m}}C_0V^{3/2} \tag{5.17}$$

follow from inequalities (4.4) and (5.14).

Substituting expressions (5.15) - (5.17) into expression (5.13) for the derivative $V(t, x, \dot{x})$, we arrive at the inequality

$$B(t,x,\dot{x})\dot{V}(t,x,\dot{x}) \le -\delta(t,x,\dot{x})V^{1/2}(t,x,\dot{x}) \tag{5.18}$$

where

$$\delta(t,x,\dot{x}) = \left(\frac{\sqrt{3}}{2\sqrt{2M}} - \frac{M_1\sqrt{5}}{m\sqrt{m}}\right)U_0 - \sqrt{\frac{5}{m}}\left(1 + \frac{M_1}{m}\right)S_0 - \frac{2\sqrt{2}}{m}C_0Q_1V^{1/2}(t,x,\dot{x}) - \frac{4\sqrt{2n}}{m\sqrt{m}}C_0V(t,x,\dot{x}) \tag{5.19}$$

The relations

$$\left|\frac{3}{4}a < \bar{A}_0\dot{x},x >\right| \le \frac{1}{2}\bar{T}_0 + \frac{9a^2}{16}\langle\bar{A}_0x,x\rangle \le \frac{1}{2}\bar{T}_0 + \frac{3b}{4}x^2$$

$$\left|\frac{3}{4}a < \bar{A}_0\dot{x},x >\right| \le \frac{3}{2}\bar{T}_0 + \frac{3a^2}{16}\langle\bar{A}_0x,x\rangle \le 2\bar{T}_0 + \frac{b}{2}x^2$$

hold by virtue of the Cauchy inequality, formula (5.2) and condition (4.2).

Using the first inequality to estimate the function $B(t, x, \dot{x})$ from below and the second to estimate the function from above, we obtain

$$0 < \frac{1}{4V}\left(2\bar{T}_0 + bx^2\right) \le B(t,x,\dot{x}) \le \frac{3}{V}\left(\bar{T}_0 + \frac{b}{2}x^2 + \frac{a}{2}\langle\bar{A}_0\dot{x},x\rangle\right) = 3 \tag{5.20}$$

Since $B(t, x, \dot{x}) > 0$, in order for the derivative $V(t, x, \dot{x})$ to be negative, it is sufficient that the expression in the brackets on the right-hand side of equality (5.18) is negative. We put

$$V(t) = V(t, x(t), \dot{x}(t)), \quad B(t) = B(t, x(t), \dot{x}(t)), \quad \delta(t) = \delta(t, x(t), \dot{x}(t)) \tag{5.21}$$

along the trajectory of the system. Taking account of relations (5.20) and (5.21), we reduce inequality (5.18) to the form

$$\dot{V}(t) \le -\frac{\delta(t)}{3} V^{1/2}(t)$$

The following assertion holds.[13]

Theorem. Suppose the condition

$$\delta(t_0) > 0 \tag{5.22}$$

is satisfied at the initial instant t_0. Then, by virtue of system (4.1), the derivative of the function V satisfies the inequality

$$\dot{V}(t) \le -\frac{\delta(t_0)}{3} V^{1/2}(t), \quad t \ge t_0 \tag{5.23}$$

It follows from the assertion of the theorem and from relations (5.6) that the value of the function V along the trajectory of system (4.1) tends to zero and the trajectory itself tends to the origin of coordinates.

Integrating inequality (5.23), we obtain the following estimate of the time of motion τ of system (4.1) from the initial state $x_0 = x(t_0)$, $\dot{x}_0 = \dot{x}(t_0)$ to the final state $x = \dot{x} = 0$

$$\tau \le \frac{6}{\delta(t_0, x_0, \dot{x}_0)} V^{1/2}(t_0, x_0, \dot{x}_0)$$

The sufficient condition for the specified final state to be reached by system (4.1)

$$U_0 > S_* + \frac{8\sqrt{2n}C}{\sqrt{3}m} V(t_0, x_0, \dot{x}_0); \quad S_* = 4\sqrt{\frac{10M}{3m}} S_0 \tag{5.24}$$

follows from relations (5.19), (5.21) and (5.22).

This condition relates the maximum admissible values of the control U_0 and the perturbations S_0, as well as the initial value of the function $V(t_0, x_0, \dot{x}_0)$ with respect to which the domain of possible initial values of the system is determined. In particular, condition (5.24) can be written in the form

$$U_0 > S_*$$

in the neighbourhood of the final state where the function $V(t, x, \dot{x})$ is small. This condition characterizes the superiority of the control forces over the perturbations, which is sufficient to attain the aim of the control.

If there are no perturbations, that is, $S_0 = 0$, the proposed control law brings system (4.1) into the final state in a finite time from any point of the domain of admissible initial states specified by the inequality

$$V(t_0, x_0, \dot{x}_0) \le \sqrt{\frac{3}{2n}} \frac{mU_0}{8C}$$

Taking account of relation (5.6), it can be asserted that this domain automatically contains the ellipsoid

$$T(t_0, x_0, \dot{x}_0) + \left(T^2(t_0, x_0, \dot{x}_0) + \frac{U_0^2}{8} x_0^2\right)^{1/2} \le \frac{mU_0}{2\sqrt{6n}C}$$

Note that the control law given by relations (5.1) - (5.3) is independent of the constants S_0 and C and the initial state $(x_0, \dot{x}_0)$, and it can therefore also be formally applied when inequality (5.24) is not satisfied. Computer modelling of the dynamics of various systems shows that the proposed law is effective far beyond the limits of sufficient conditions (5.24). This is explained by the fact that condition (5.24) guarantees a monotonic decrease in the function V along the trajectory of system (4.1) which is controlled by law (5.1) - (5.3). However, the function V can tend to zero non-monotonically, and, in this case, the trajectories of the system will arrive at the final state as before.

6. Modelling of the dynamics of a double pendulum

We will now present the results of numerical modelling of the controlled motions of a plane double pendulum. It is assumed in the modelling that the links of the pendulum are homogeneous rods, the hinges are located at the ends of the links and that a load (a point mass) is fixed to the end of the second link.

The equations of motion of this mechanical system are written in the form

$$A(q)\ddot{q} = u + G(q, \dot{q}) \tag{6.1}$$

where

$$A(q) = \begin{Vmatrix} 2\left(\frac{m_1}{3} + \xi_1\right) l_1^2 & \frac{\xi_4}{2} l_1 l_2 \cos(q_1 - q_2) \\ \frac{\xi_4}{2} l_1 l_2 \cos(q_1 - q_2) & \frac{2\xi_3}{3} l_2^2 \end{Vmatrix}$$

$$G(q,\dot{q}) = \begin{Vmatrix} -\frac{\xi_4}{2} l_1 l_2 \sin(q_1 - q_2)\dot{q}_2^2 & -\left(\frac{m_1}{2} + \xi_1\right) g l_1 \sin(q_1) \\ \frac{\xi_4}{2} l_1 l_2 \sin(q_1 - q_2)\dot{q}_1^2 & -\frac{\xi_2}{2} g l_2 \sin(q_2) \end{Vmatrix}$$

$$\xi_s = m_2 + s m_3, \quad s = 1,2,3,4$$

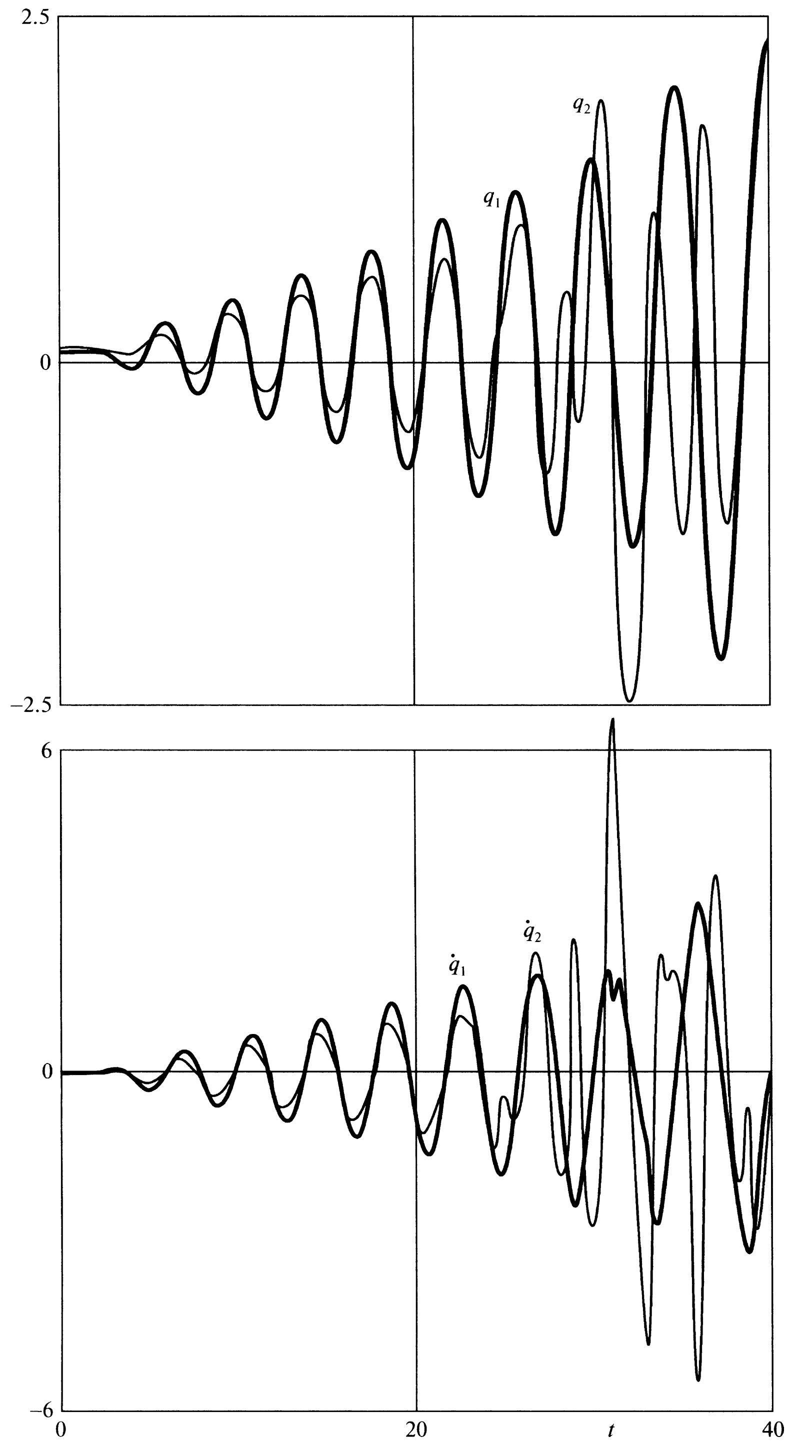

Fig. 1.

Here q_1 and q_2 are the angles of the inclinations of the links of the pendulum from the vertical, m_1 and m_2 and l_1 and l_2 are the masses and lengths of the links, m_3 is the mass of the load and g is the acceleration due to gravity.

It is required to bring the pendulum into the state

$$q_1 = q_2 = 3\pi/4 \ \text{rad}, \quad \dot{q}_1 = \dot{q}_2 = 0 \tag{6.2}$$

Calculations were carried out for the following values of the parameters

$$m_1 = 10 \ \text{kg}, \quad m_2 = 5 \ \text{kg}, \quad m_3 = 3 \ \text{kg}, \quad l_1 = 0.8 \ \text{m}, \quad l_2 = 0.5 \ \text{m}$$

The maximum admissible value of the norm of the control moment vector was chosen to be equal to $u_0 = 10$ Nm. In this case, half the resources of the control were employed in moving the unperturbed system along the nominal trajectory and the other half in bringing the pendulum into the nominal trajectory and countering the deviations from it.

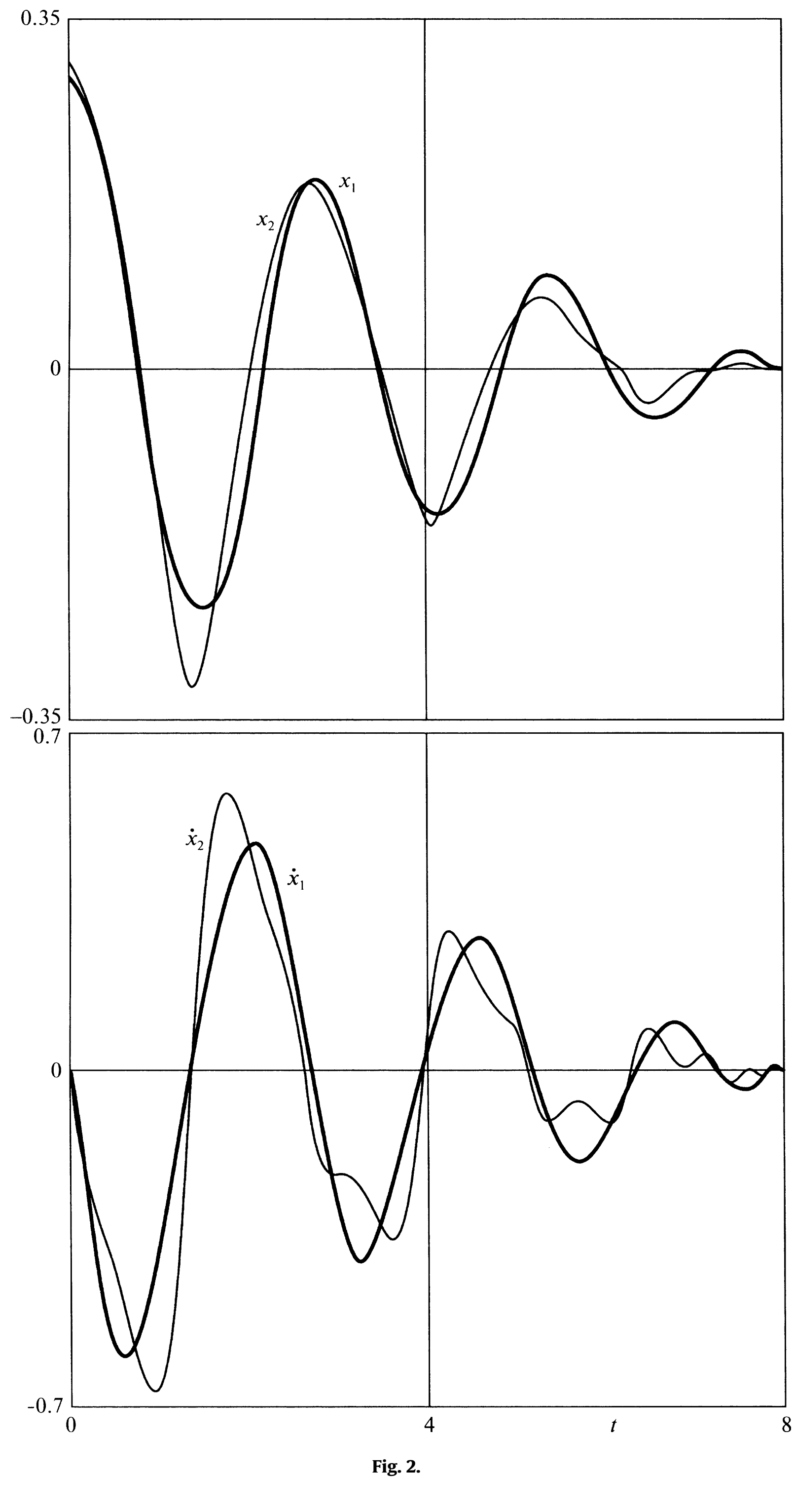

Fig. 2.

The results of the modelling of the first stage of the motion of the pendulum are not presented here. It is assumed that, at the end of this stage, the pendulum is in the state

$$q_1 = 0.37 \text{ rad}, \quad q_2 = 0.4 \text{ rad}, \quad \dot{q}_1 = \dot{q}_2 = 0 \tag{6.3}$$

In order to construct the nominal trajectory in the second stage of the motion, the pendulum was transferred using a numerical experiment from the state (6.2) in the neighbourhood of the zero state of rest using brake action, that is, using control moments directed against the angular velocity vector

$$u = -\frac{u_0}{2|\dot{q}|}\dot{q}, \quad \dot{q} \neq 0$$

The resulting trajectory of the motion, taken in reverse time, was chosen as the nominal trajectory for the second stage of the motion, and the state

$$q_1 = 0.07 \text{ rad}, \quad q_2 = 0.08 \text{ rad}, \quad \dot{q}_1 = \dot{q}_2 = 0 \tag{6.4}$$

in which the pendulum was stopped, in the numerical experiment was chosen as the initial point of the nominal trajectory. The initial deviations from the nominal trajectory which, according to the algorithm are equal to the difference between state vectors (6.3) and (6.4), were found to be as follows:

$$q_1 = 0.30 \text{ rad}, \quad q_2 = 0.32 \text{ rad}, \quad \dot{q}_1 = \dot{q}_2 = 0$$

In the second stage, the pendulum was brought into the nominal trajectory after which, by moving along it, it arrived in the final state (6.2).

All the figures presented below refer to the second stage of the motion. The thick lines describe the behaviour of the first link and the thin lines describe the behaviour of the second link. The time dependences of the angular coordinates of the links are shown in the upper part of Fig. 1 and the time dependences of the angular velocities are shown in the lower part. The overall time of the motion of the system up to the final state was about 40s.

Graphs of the deviations of the links from the nominal trajectory with respect to the angular coordinates are shown in the upper part of Fig. 2, and graphs of the deviations of the links with respect to the angular velocities are shown in the lower part. It is seen that the pendulum reaches the nominal trajectory in about 8s and subsequently does not leave it until arriving at the final state.

The investigation show that the approach proposed earlier[14] is not only applicable to the Lagrangian systems (1.1) but also to systems of the form (4.1). The considerable difference between these systems lies in the fact that the theorem concerning the change in the kinetic energy is not satisfied in the form which holds in the case of Lagrangian systems. This fact affects the choice of Lyapunov's function for Eqs. (4.1). Nevertheless, control law (5.1) - (5.3), which is similar to that proposed earlier[14] for Lagrangian systems, also turns out to be effective in the case of deviation equations (4.1). Hence, the approach described above enables one to construct a bounded control for a mechanical system with parameters which brings the system from an arbitrary initial state into a specified final state in a finite time, under the assumption that potential forces which are greater in magnitude than the control forces act on the system. The results of the modelling of the dynamics of a double pendulum controlled by small forces presented above serve as an illustration.

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